Pure Type Systems

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Introduction

Some history of PL theory

- [Church, 1936] Untyped λ-calculus

$$\text{Terms} \qquad M, N \quad ::= \quad x \mid \lambda x.M \mid M\,N$$

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- [Church, 1936] Untyped λ-calculus
- [Church, 1940] Simply-typed λ-calculus (STLC)

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\begin{array}{llcl}
\text{Terms} & M, N & ::= & x \mid \lambda x{:}A.M \mid M\,N \\
\text{Types} & A, B & ::= & \mathbf{b} \mid A \to B \\
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\end{array}
$$

$$
\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B} \qquad \frac{\Gamma \vdash A : \mathsf{type} \quad \Gamma \vdash B : \mathsf{type}}{\Gamma \vdash A \to B : \mathsf{type}}
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Some history of PL theory

- [Church, 1936] Untyped λ-calculus
- [Church, 1940] Simply-typed λ-calculus (STLC)
- [Girard, 1972, Reynolds, 1974] Polymorphic λ-calculus (PLC)

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\frac{\Gamma, \alpha \vdash M : A}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.A}
\qquad
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- [Renardel de Lavalette, 1987] Higher-order kinds (POLYREC)

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\end{array}$$

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  - [Harper *et al.*, 1987] Logical framework LF

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\frac{\Gamma, x{:}A \vdash B : K}{\Gamma \vdash \lambda x{:}A.B : \Pi x{:}A.K}
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Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N$
Types	A, B	$::=$	$\mathbf{b} \mid A \to B \mid \lambda x{:}A.B \mid A\,M$
Kinds	K, L	$::=$	$\star \mid \Pi x{:}A.K$
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$$\frac{\Gamma, x{:}A \vdash B : K}{\Gamma \vdash \lambda x{:}A.B : \Pi x{:}A.K} \qquad \frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\,M : [M/x]K} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \square}{\Gamma \vdash \Pi x{:}A.K : \square}$$

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- [Coquand and Huet, 1988] Calculus of constructions (CoC)

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\begin{array}{llcl}
\text{Terms} & M, N & ::= & x \mid \lambda x{:}A.M \mid M\,N \mid \Lambda\alpha{:}K.M \mid M\,A \\
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\text{Kinds} & K, L & ::= & \star \mid (\alpha{:}K) \Rightarrow L \mid \Pi x{:}A.K \\
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\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash (x{:}A) \to B : \star}
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$$

INTRODUCTION

UNIFIED FUNCTION SPACES

[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : (x{:}A) \to B} \qquad \frac{\Gamma \vdash M : (x{:}A) \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash (x{:}A) \to B : \star}$$

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Introduction

Unified function spaces

[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma, \alpha{:}K \vdash M : A}{\Gamma \vdash \Lambda\alpha{:}K.M : \Pi\alpha{:}K.A} \qquad \frac{\Gamma \vdash M : \Pi\alpha{:}K.B \quad \Gamma \vdash A : K}{\Gamma \vdash M\,A : [A/\alpha]B} \qquad \frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi\alpha{:}K.A : \star}$$

$$\frac{\Gamma, x{:}A \vdash B : K}{\Gamma \vdash \lambda x{:}A.B : \Pi x{:}A.K} \qquad \frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\,M : [M/x]K} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma, \alpha{:}K \vdash A : L}{\Gamma \vdash \Lambda\alpha{:}K.A : \Pi\alpha{:}K.L} \qquad \frac{\Gamma \vdash A : \Pi\alpha{:}L.K \quad \Gamma \vdash B : L}{\Gamma \vdash A\,B : [B/\alpha]K} \qquad \frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi\alpha{:}K.L : \Box}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma, \alpha{:}K \vdash M : A}{\Gamma \vdash \lambda \alpha{:}K.M : \Pi \alpha{:}K.A} \qquad \frac{\Gamma \vdash M : \Pi \alpha{:}K.B \quad \Gamma \vdash A : K}{\Gamma \vdash M\, A : [A/\alpha]B} \qquad \frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma, x{:}A \vdash B : K}{\Gamma \vdash \lambda x{:}A.B : \Pi x{:}A.K} \qquad \frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\, M : [M/x]K} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma, \alpha{:}K \vdash A : L}{\Gamma \vdash \lambda \alpha{:}K.A : \Pi \alpha{:}K.L} \qquad \frac{\Gamma \vdash A : \Pi \alpha{:}L.K \quad \Gamma \vdash B : L}{\Gamma \vdash A\, B : [B/\alpha]K} \qquad \frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma, \alpha{:}K \vdash M : A}{\Gamma \vdash \lambda \alpha{:}K.M : \Pi \alpha{:}K.A} \qquad \frac{\Gamma \vdash M : \Pi \alpha{:}K.B \quad \Gamma \vdash A : K}{\Gamma \vdash M\,A : [A/\alpha]B} \qquad \frac{\Gamma \vdash K : \square \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma, x{:}A \vdash B : K}{\Gamma \vdash \lambda x{:}A.B : \Pi x{:}A.K} \qquad \frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\,M : [M/x]K} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \square}{\Gamma \vdash \Pi x{:}A.K : \square}$$

$$\frac{\Gamma, \alpha{:}K \vdash A : L}{\Gamma \vdash \lambda \alpha{:}K.A : \Pi \alpha{:}K.L} \qquad \frac{\Gamma \vdash A : \Pi \alpha{:}L.K \quad \Gamma \vdash B : L}{\Gamma \vdash A\,B : [B/\alpha]K} \qquad \frac{\Gamma \vdash K : \square \quad \Gamma, \alpha{:}K \vdash L : \square}{\Gamma \vdash \Pi \alpha{:}K.L : \square}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$$

$$\frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash M : \Pi \alpha{:}K.B \quad \Gamma \vdash A : K}{\Gamma \vdash M\,A : [A/\alpha]B}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\,M : [M/x]K}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash A : \Pi \alpha{:}L.K \quad \Gamma \vdash B : L}{\Gamma \vdash A\,B : [B/\alpha]K}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$$

$$\frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash M : \Pi \alpha{:}K.B \quad \Gamma \vdash A : K}{\Gamma \vdash M\,A : [A/\alpha]B}$$

$$\frac{\Gamma \vdash A : \Pi x{:}B.K \quad \Gamma \vdash M : B}{\Gamma \vdash A\,M : [M/x]K}$$

$$\frac{\Gamma \vdash A : \Pi \alpha{:}L.K \quad \Gamma \vdash B : L}{\Gamma \vdash A\,B : [B/\alpha]K}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$$

$$\boxed{\frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Lambda \alpha{:}K.M \mid M\,A$
Types	A, B	$::=$	$\mathbf{b} \mid (x{:}A) \to B \mid \forall \alpha{:}K.A \mid \lambda x{:}A.B \mid A\,M \mid \Lambda \alpha{:}K.A \mid A\,B$
Kinds	K, L	$::=$	$\star \mid (\alpha{:}K) \Rightarrow L \mid \Pi x{:}A.K$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A \mid \Gamma, \alpha{:}K$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Lambda\alpha{:}K.M \mid M\,A$
Types	A, B	$::=$	$\mathbf{b} \mid (x{:}A) \to B \mid \forall\alpha{:}K.A \mid \lambda x{:}A.B \mid A\,M \mid \Lambda\alpha{:}K.A \mid A\,B$
Kinds	K, L	$::=$	$\star \mid (\alpha{:}K) \Rightarrow L \mid \Pi x{:}A.K$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A \mid \Gamma, \alpha{:}K$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N$
Types	A, B	$::=$	$\mathbf{b} \mid (x{:}A) \to B \mid \forall \alpha{:}K.A$
Kinds	K, L	$::=$	$\star \mid (\alpha{:}K) \Rightarrow L \mid \Pi x{:}A.K$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A \mid \Gamma, \alpha{:}K$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N$
Types	A, B	$::=$	$\mathbf{b}$
Kinds	K, L	$::=$	$\star \mid \Pi x{:}A.K$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A \mid \Gamma, \alpha{:}K$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Terms	M, N	$::=$	$x \mid \lambda x{:}A.M \mid M\,N$
Types	A, B	$::=$	$\mathbf{b}$
Kinds	K, L	$::=$	$\star \mid \Pi x{:}A.K$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid \star$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \square \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \square}{\Gamma \vdash \Pi x{:}A.K : \square}$$

$$\frac{\Gamma \vdash K : \square \quad \Gamma, \alpha{:}K \vdash L : \square}{\Gamma \vdash \Pi \alpha{:}K.L : \square}$$

Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid \star$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid \star$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Sorts	s	$::=$	$\star \mid \Box$
Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid s$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.B : \star}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash A : \star}{\Gamma \vdash \Pi \alpha{:}K.A : \star}$$

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x{:}A \vdash K : \Box}{\Gamma \vdash \Pi x{:}A.K : \Box}$$

$$\frac{\Gamma \vdash K : \Box \quad \Gamma, \alpha{:}K \vdash L : \Box}{\Gamma \vdash \Pi \alpha{:}K.L : \Box}$$

Sorts	s	$::=$	$\star \mid \Box$
Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid s$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$$

$$\frac{\Gamma \vdash K : s_1 \quad \Gamma, \alpha{:}K \vdash A : s_2}{\Gamma \vdash \Pi \alpha{:}K.A : s_2}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash K : s_2}{\Gamma \vdash \Pi x{:}A.K : s_2}$$

$$\frac{\Gamma \vdash K : s_1 \quad \Gamma, \alpha{:}K \vdash L : s_2}{\Gamma \vdash \Pi \alpha{:}K.L : s_2}$$

Sorts	s	$::=$	$\star \mid \square$
Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid s$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \boxed{\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}}$$

Sorts	s	$::=$	$\star \mid \square$
Expressions	M, N, A, B	$::=$	$x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid s$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

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UNIFIED FUNCTION SPACES

[Berardi, 1988; Terlouw, 1988; Barendregt, 1991] Generalized type systems

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [N/x]B} \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$$

$$\begin{array}{lrcl} \text{Sorts} & s & ::= & \star \mid \square \\ \text{Expressions} & M, N, A, B & ::= & x \mid \lambda x{:}A.M \mid M\,N \mid \Pi x{:}A.B \mid s \\ \text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x{:}A \end{array}$$

This gives a concise encoding of CoC
What about STLC, PLC, LF, etc.?

Pure type systems
Definition

PTS is a typed λ-calculus parameterized with:

1. A set of *sorts* $s \in \mathcal{S}$
2. A relation of *axioms* $s_1 : s_2 \in \mathcal{A}$
3. A relation of *rules* $(s_1, s_2, s_3) \in \mathcal{R}$

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$$\frac{\vdash \Gamma \quad s_1 : s_2 \in \mathcal{A}}{\Gamma \vdash s_1 : s_2} \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2 \quad (s_1, s_2, s_3) \in \mathcal{R}}{\Gamma \vdash \Pi x{:}A.B : s_3}$$

$$\frac{\Gamma \vdash \Pi x{:}A.B : s \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : [M/x]B}$$

$$\frac{\vdash \Gamma \quad \Gamma \vdash A : s}{\vdash \Gamma, x{:}A}$$

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EXAMPLES

λ-CUBE [BARENDREGT, 1991]

8 languages related by inclusion. All have the same sorts and axioms:

- $\mathcal{S} = \{\star, \Box\}$
- $\mathcal{A} = \{\star : \Box\}$
- Rules all have the form (s_1, s_2, s_2), abbreviated (s_1, s_2).
  All languages have the rule $(\star, \star)$, and some subset of $\{(\star, \Box), (\Box, \star), (\Box, \Box)\}$.

Examples

λ-cube [Barendregt, 1991]

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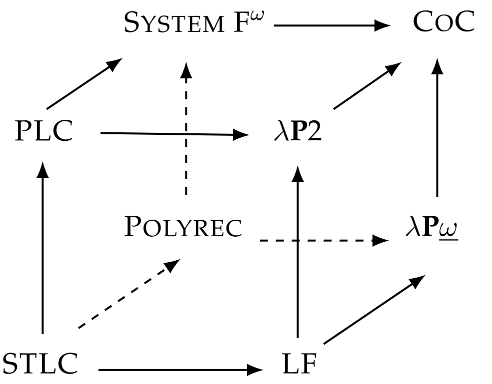
STLC	$(\star, \star)$			
PLC	$(\star, \star)$	$(\Box, \star)$		
PolyRec	$(\star, \star)$		$(\Box, \Box)$	
System F^ω	$(\star, \star)$	$(\Box, \star)$	$(\Box, \Box)$	
LF	$(\star, \star)$			$(\star, \Box)$
λ**P2**	$(\star, \star)$	$(\Box, \star)$		$(\star, \Box)$
$\lambda\mathbf{P}\underline{\omega}$	$(\star, \star)$		$(\Box, \Box)$	$(\star, \Box)$
CoC	$(\star, \star)$	$(\Box, \star)$	$(\Box, \Box)$	$(\star, \Box)$



Examples

Martin-Löf Type Theory

PTS specification for MLTT

- $\mathcal{S} = \{\mathsf{Set}_i \mid i \in \mathbb{N}\}$
- $\mathcal{A} = \{\mathsf{Set}_i : \mathsf{Set}_j \mid i \leq j\}$
- $\mathcal{R} = \{\mathsf{Set}_i, \mathsf{Set}_j, \mathsf{Set}_k \mid i, j \leq k\}$

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EXAMPLES

MARTIN-LÖF TYPE THEORY

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EXAMPLES

MARTIN-LÖF TYPE THEORY

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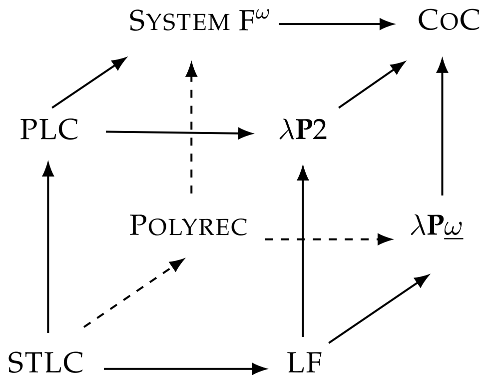
COC	$(\star,\star)$	$(\square,\star)$	$(\square,\square)$	$(\star,\square)$



PROPERTIES OF PTSs

BASIC PROPERTIES

Evaluation

This standard small-step semantics works with all PTSs

$$\frac{}{\Gamma \vdash (\lambda x{:}A.M)\ N \Longrightarrow_\beta [N/x]M} \qquad \frac{\Gamma \vdash M \Longrightarrow_\beta M'}{\Gamma \vdash M\ N \Longrightarrow_\beta M'\ N} \qquad \frac{\Gamma \vdash N \Longrightarrow_\beta N'}{\Gamma \vdash M\ N \Longrightarrow_\beta M\ N'}$$

Lemma (Substitution property)

If $\Gamma, x{:}A \vdash M : B$ *and* $\Gamma \vdash N : A$*, then* $\Gamma \vdash [N/x]M : [N/x]B$

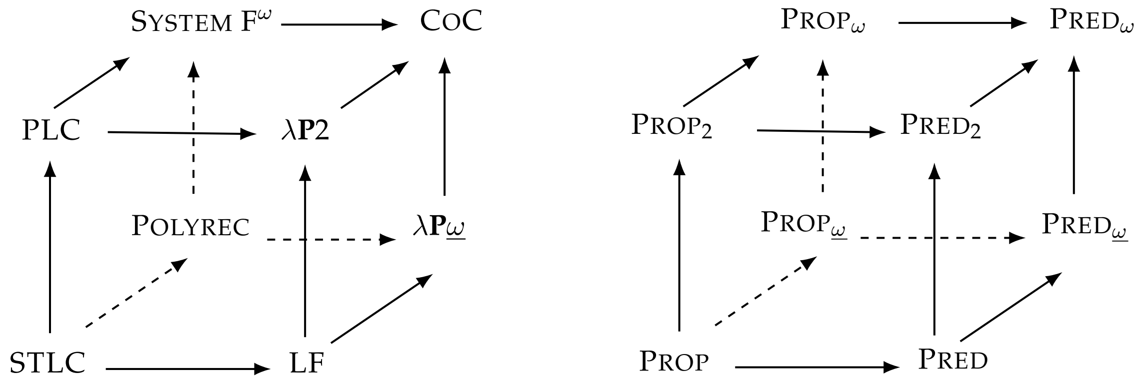
Theorem (Subject reduction)

If $\Gamma \vdash M : A$ *and* $\Gamma \vdash M \Longrightarrow_\beta M'$*, then* $\Gamma \vdash M' : A$*.*

PROPERTIES OF PTSS
CURRY-HOWARD ISOMORPHISMS FOR THE λ-CUBE

Theorem (Barendregt, 1991)

*Every system in the λ-cube admits a Curry-Howard isomorphism, and the corresponding logics form an analogous **L**-cube.*



Properties of PTSs

Normalization for predicative PTSs

Definition (Predicativity)

A PTS is *predicative* if there is a partial order $\preceq$ over $\mathcal{S}$ such that:

1. If $s_1 : s_2 \in \mathcal{A}$, then $s_1 \preceq s_2$
2. If $(s_1, s_2, s_3) \in \mathcal{R}$, then $s_1 \preceq s_3$ and $s_2 \preceq s_3$

Theorem (Strong normalization for predicative PTSs [Fridlender and Pagano, 2015])

In a predicative PTS, if $\Gamma \vdash M : A$*, then* M *is strongly normalizing.*

Proof.

Generalize NbE for MLTT □

Extensions of PTSs

- Pairs and Σ-types

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2 \quad (s_1, s_2, s_3) \in \mathcal{R}}{\Gamma \vdash \Sigma x{:}A.B : s_3}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : [M/x]B}{\Gamma \vdash \langle M, N \rangle : \Sigma x{:}A.B} \qquad \frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_1 M : A} \qquad \frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_2 M : [\pi_1 M/x]B}$$

- Constants and signatures

$$\frac{\vdash \mathsf{sig} \quad \cdot \vdash_{\mathsf{sig}} A : s \quad \mathbf{c} \notin \mathsf{sig}}{\vdash \mathsf{sig}, \mathbf{c}{:}A} \qquad \frac{\vdash_{\mathsf{sig}} \Gamma \quad (\mathbf{c}{:}A) \in \mathsf{sig}}{\Gamma \vdash_{\mathsf{sig}} \mathbf{c} : A}$$

Extensions of PTSs

Definitions

[Severi and Poll, 1994, Stone and Harper, 2006] Singleton types and let-definitions

$$\frac{\Gamma \vdash A : s \quad \Gamma \vdash M : A}{\Gamma \vdash \mathsf{S}_A(M) : s} \qquad \frac{\Gamma \vdash M : A \quad \Gamma, x{:}\mathsf{S}_A(M) \vdash N : B}{\Gamma \vdash \mathsf{let}\ x = M{:}A\ \mathsf{in}\ N : [M/x]B} \qquad \frac{\Gamma \vdash M : \mathsf{S}_A(N)}{\Gamma \vdash M \Longrightarrow_{\beta\delta} N : A}$$

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[Severi and Poll, 1994, Stone and Harper, 2006] Singleton types and let-definitions

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[Barthe, 1995] PTS with quotient types and PTS with subset types

Theorem

Let $\lambda\mathcal{S}$ be a PTS containing HOL. TFAE:

1. *The extension of $\lambda\mathcal{S}$ with definitions is strongly normalizing*
2. *The extension of $\lambda\mathcal{S}$ with quotient types is strongly normalizing*
3. *The extension of $\lambda\mathcal{S}$ with subset types is strongly normalizing*

Extensions of PTSs
Some more extensions

- [Courant, 1997] Module calculus
- [Borghuis, 1998] Modal PTS
- [Zwanenburg, 1999] Subtyping
- [Barthe et al., 2003] Pattern matching
- [Severi and de Vries, 2012] Corecursion on streams
- [Roux and van Doorn, 2014] Construct larger PTSs while preserving normalization
  - $\mathcal{P} + \mathcal{Q}$. Disjoint sum of PTSs $\mathcal{P}$ and $\mathcal{Q}$
  - $\forall \mathcal{P}.\mathcal{Q}$. Also allows forming $\mathcal{Q}$-types by quantifying over $\mathcal{P}$-types
  - $\mathcal{P}_{\text{POLY}}$. Close $\mathcal{R}_{\mathcal{P}}$ with $(s_1, s_2, s^*_{s_1,s_2})$ for all $s_1, s_2 \in \mathcal{S}_{\mathcal{P}}$
- [Yang and Oliveira, 2019] General recursion and iso-types

Conclusion

Pure type systems

- Express a lot of type systems
- Can be extended in many ways
- Prove properties about several type systems at once

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